Solution Guide: Pearls in Graph Theory
Sections 10.1, 10.2, and 10.3

Austin Ulrigg

August 1, 2024

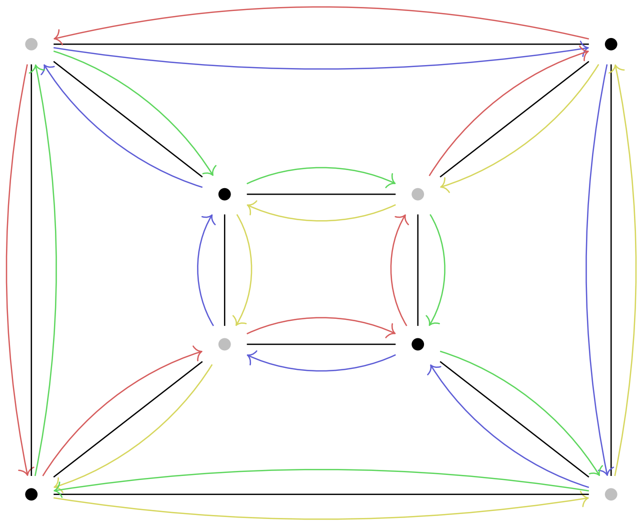
Contents

1 Section 10.1: Rotations of Graphs

Exercise 10.1.1

Problem: Find a rotation of the cube graph of Figure 1.2.5 inducing four circuits, each of length 6.

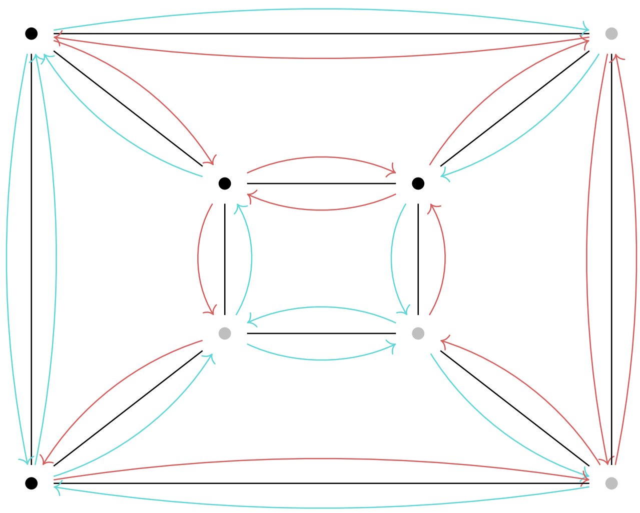
Solution:



Exercise 10.1.2

Problem: Find a rotation of the cube graph inducing two circuits.

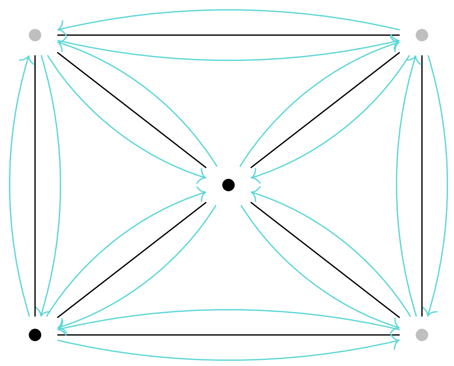
Solution:



Exercise 10.1.3

Problem: Find a circular rotation of the wheel graph W_4.

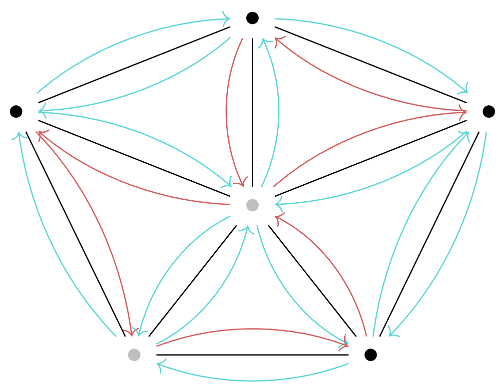
Solution:



Exercise 10.1.4

Problem: Find a rotation of the wheel graph W_5 inducing 2 circuits.

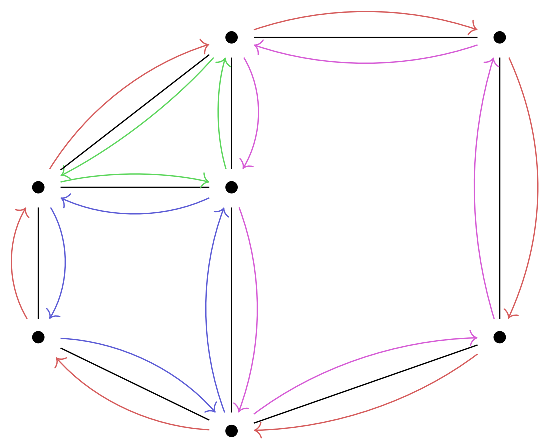
Solution:



Exercise 10.1.5

Problem: Find a connected graph G with a rotation p such that there are four induced circuits and their lengths are 3, 4, 5, and 6.

Solution:



Exercise 10.1.6

Problem: Prove it is not possible to find a graph G with a rotation p such that the lengths of the five induced circuits are 3, 4, 5, 5, and 6.

Solution: If a rotation p of a graph G, say with m edges, induced 5 circuits, then every edge in G must've been traversed twice in both directions over all of the circuits. However, $3+4+5+5+6 = 23 \neq 2m$ is odd and cannot equal $2m$, so such a graph/rotation cannot exist.

Exercise 10.1.7

Problem: How many different rotations does a cubic graph with $2n$ vertices have?

Solution: All vertices in a cubic graph are degree 3, so there are $(3-1)! = 2! = 2$ rotations per vertex, and thus $2^{(2n)}$ total rotations of the graph.

Exercise 10.1.8

Problem: Suppose that a graph G has p vertices with degrees $d_1, d_2, \dots d_p$. How many different rotations does G have?

Solution: There are $(d_1 - 1)!(d_2 - 1)! \dots (d_p - 1)!$ total different rotations of G.

Exercise 10.1.9

Problem: List the circuits induced by the following rotation of $K_{3,3}$.

0.	1	3	5
1.	2	4	0
2.	1	5	3
3.	2	4	0
4.	1	3	5
5.	2	0	4

Solution: The circuits can be read off the scheme to be

$$1 \to 0 \to 3 \to 2 \to 1 \to 4 \to 3 \to 0 \to 5 \to 4 \to 1 \to 0 \to \dots$$

$$0 \to 1 \to 2 \to 5 \to 0 \to 1 \to \dots$$

$$2 \to 3 \to 4 \to 5 \to 2 \to 3 \to \dots$$

This is all circuits since their combined length is 18 and there are 9 edges in $K_{3,3}$ so they were all traversed twice in both directions.

Exercise 10.1.10

Problem: List the circuits induced by the following rotation of K_5.

0.	1	2	4	3
1.	0	2	4	3
2.	0	1	4	3
3.	0	1	4	2
4.	0	2	3	1

Solution: The circuits can be read off the scheme to be

$$0 \to 1 \to 2 \to 4 \to 3 \to 2 \to 0 \to 4 \to 2 \to 3 \to 0 \to 1 \to \dots$$

$$1 \to 0 \to 2 \to 1 \to 4 \to 0 \to 3 \to 1 \to 0 \to \dots$$

$$3 \to 4 \to 1 \to 3 \to 4 \to \dots$$

This is all circuits since their combined length is 20 and there are 10 edges in K_5 so they were all traversed twice in both directions.

Exercise 10.1.11

Problem: Find a circular rotation of $K_{3,3}$.

Solution: A circular rotation of $K_{3,3}$ would induce a circuit of length 18, the following scheme gives one.

0.	3	5	1
1.	0	4	2
2.	5	3	1
3.	2	0	4
4.	1	5	3
5.	4	2	3

And the circuit is,

$$0 \to 1 \to 4 \to 5 \to 2 \to 3 \to 0 \to 5 \to 4 \to 3 \to 2 \to 1 \to 0 \to 3 \to 4 \to 1 \to 2 \to 5 \to 0 \to 1 \to \dots$$

2 Section 10.2: Planar Graphs Revisited

Exercise 10.2.1

Problem: Show that the graph G of Figure 8.1.2 is planar under the definition of Chapter 10.

Solution: The graph has 8 vertices and 18 edges so under the definition of Chapter 10 it is planar if there exists a rotation p which induces 12 circuits. Such a rotation is given by the following scheme,

1.	5	4	2	3	7	8
2.	3	1	4			
3.	5	7	1	2	4	
4.	5	3	2	1		
5.	8	6	7	3	4	1
6.	7	5	8			
7.	6	8	1	3	5	
8.	6	5	1	7		

Exercise 10.2.2

Problem: Prove that a tree is planar under the definition of Chapter 10.

Solution: Either the tree is a spanning tree, or it has K disconnected components, each being spanning trees. A spanning tree has p vertices and $p-1$ edges and always has only one circuit under any given rotation p. Therefore, for any rotation p we have $p - q + r(p) = p - (p-1) + 1 = 2$ proving that all spanning trees are planar under the definition of Chapter 10, and hence that all the connected components of any tree are planar under the definition of Chapter 10 which completes the proof.

Exercise 10.2.3

Problem: Prove or disprove: Given a graph G, if every subgraph of G that is different from G is planar, then G is planar.

Solution: The statement is false. It is easy to see that removing any edge from $K_{3,3}$ makes the resulting graph embeddable in the plane. Therefore any $K_{3,3} - e$ is planar, and if you remove multiple edges, simply taking the planar embedding of $K_{3,3} - e$ and removing the same edges from it will leave you with a still planar embedding. Removing any vertice(s) will remove multiple edges, and hence also have an embedding. Therefore, any subgraph of $K_{3,3}$ that is different from $K_{3,3}$ is planar by the Chapter 8 definition, which Theorem 10.2.9 then implies that any subgraph of $K_{3,3}$ is also planar by the Chapter 10 definition, however, $K_{3,3}$ is not planar as demonstrated by the example in the book.

Exercise 10.2.4

Problem: Prove Theorem 10.2.4.

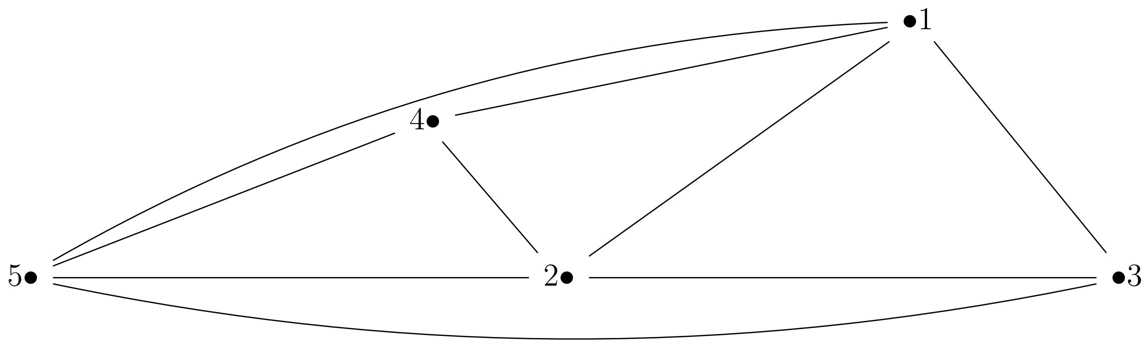
Solution: If K_5 is planar under the Chapter 10 definition then there exists a rotation p inducing $r(p)$ circuits such that $5 - 10 + r(p) = 2 \implies r(p) = 7$. Because K_5 has 10 edges, any rotation of it must create circuits of lengths that sum to 20 in order to traverse all the edges in both directions. This implies that a planar rotation of K_5 induces circuits which have an average length of $\frac{20}{7} < 3$, which implies there exists a circuit of length 2, which is not possible.

Exercise 10.2.5

Problem: Find a planar rotation of $K_5 - e$.

Solution: Removing any edge of K_5 is isomorphic to removing the edge between v_3 and v_4 without loss of generality. This scheme gives a planar rotation of $K_5 - e$ which is pictured below.

1.	2	3	5	4
2.	1	4	5	3
3.	1	2	5	
4.	1	5	2	
5.	1	3	2	4

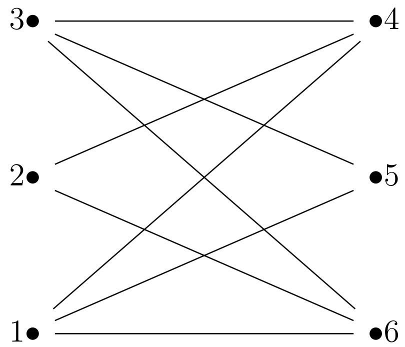


Exercise 10.2.6

Problem: Find a planar rotation of $K_{3,3} - e$.

Solution: Removing any edge from $K_{3,3}$ is isomorphic to removing the edge between v_2 and v_5 without loss of generality. $K_{3,3} - e$ has 6 vertices and 8 edges thus a planar rotation would induce 5 circuits. This scheme gives a planar rotation of $K_{3,3} - e$ which is pictured below.

1.	6	4	5
2.	6	4	
3.	6	5	4
4.	3	1	2
5.	1	3	
6.	1	3	2



Exercise 10.2.7

Problem: Suppose that p is a rotation of K_n. Is $r(p)$ odd or even?

Solution: Because K_n is a connected graph with n vertices and $\frac{n(n-1)}{2}$ edges we know for any rotation p of K_n that by theorem 10.1.2 $n - \frac{n(n-1)}{2} + r(p)$ is even. Thus,

$$n - \frac{n(n-1)}{2} + r(p) = 2m \quad m \in \mathbb{Z}$$

$$2n - n(n-1) + 2r(p) = 4m$$

$$n(3-n) + 2r(p) = 4m$$

$$n(3-n)+2r(p) \equiv 0 \mod 4$$

If n is even it reduces to 0 or $2 \mod 4$, therefore if n is even, either,

$$2r(p) \equiv 0 \mod 4$$

which gives $r(p)$ is even or,

$$2(3-2)+2r(p) \equiv 0 \mod 4$$

$$2+2r(p) \equiv 0 \mod 4$$

which gives $r(p)$ is odd
Otherwise if n is odd it reduces to 1 or $3 \mod 4$, therefore if n is odd, either,

$$1(3-1)+2r(p) \equiv 0 \mod 4$$

$$2+2r(p) \equiv 0 \mod 4$$

which gives $r(p)$ is odd or,

$$3(3-3)+2r(p) \equiv 0 \mod 4$$

$$2r(p) \equiv 0 \mod 4$$

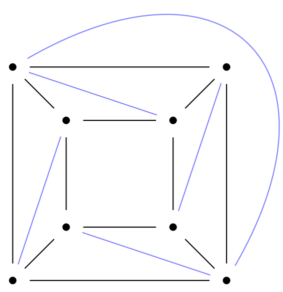
which gives $r(p)$ is even. Thus, if $n \equiv 0 \mod 4$ or $n \equiv 3 \mod 4$ then $r(p)$ is even, and if $n \equiv 1 \mod 4$ or $n \equiv 2 \mod 4$ then $r(p)$ is odd.

3 Section 10.3: The Genus of a Graph

Exercise 10.3.1

Problem: Without adding any vertices, what is the maximum number of edges that can be added to the cube graph of Figure 1.2.5 so that the resulting graph is still planar?

Solution: One can see from the following planar embedding that adding 6 edges is possible.



Therefore, at least 6 edges can be added to the cube graph so that the resulting graph is still planar. However, adding 7 edges is not possible. For the cube graph plus 7 edges to be planar that would require the existence of a rotation p such that

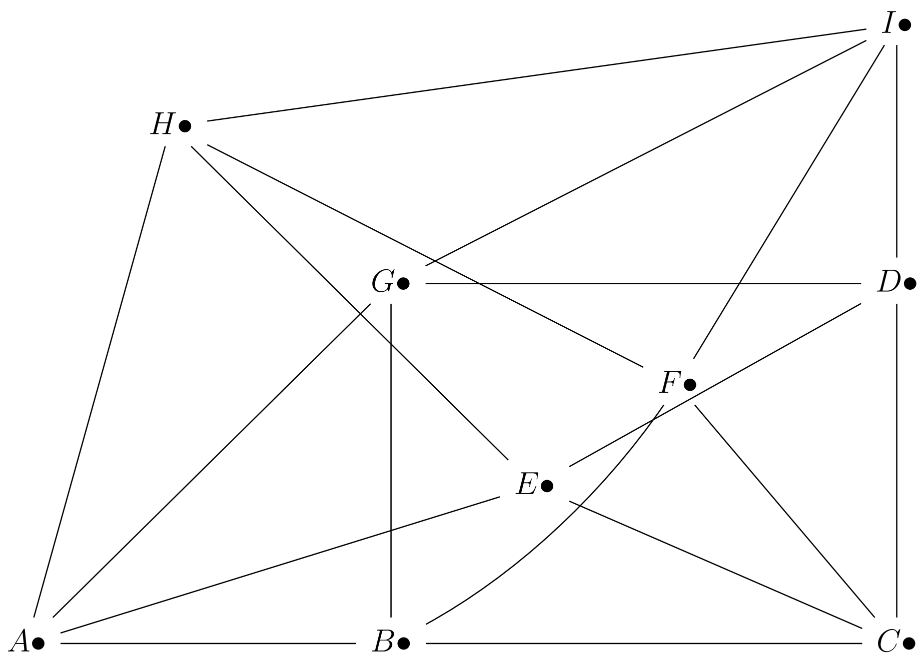
$$8-19+r(p)=2 \implies r(p)=13$$

However, in such a graph there are 19 edges and thus in total all induced circuits must traverse 38 directed edges, which implies that the average length of our 13 induced circuits is $\frac{38}{13} < 3$, however in the cube graph all vertices already have degree higher than 2, and adding edges only raises the degree. Therefore, it is not possible to have a circuit of length less than 3, and the maximum amount of edges that can be added so that the resulting graph is still planar is 6.

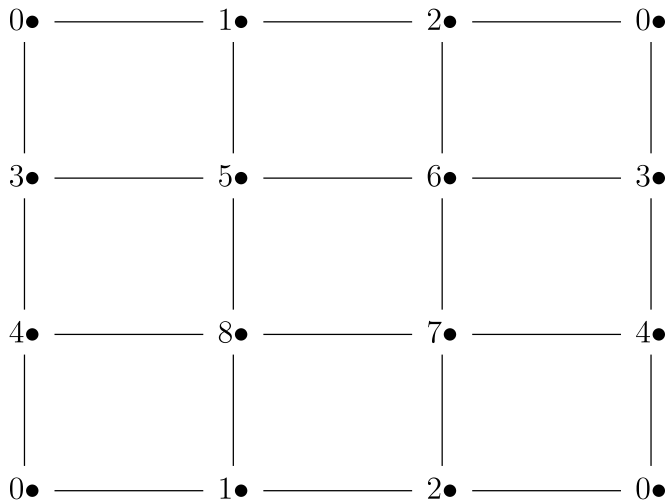
Exercise 10.3.2

Problem: Verify that the graphs pictured in Figure 8.4.12 and 10.3.8 are isomorphic.

Solution: After labeling the vertices in Figure 8.4.12 we have the following graph:



The graph in Figure 10.3.8 is:



We define our isomorphism f as

$$0 \mapsto I, 1 \mapsto D, 2 \mapsto G, 3 \mapsto F, 4 \mapsto H, 5 \mapsto C, 6 \mapsto B, 7 \mapsto A, 8 \mapsto E$$

And from this we have that v_i is connected to v_j in Figure 10.3.8 if and only if $f(v_i)$ is connected to $f(v_j)$ is Figure 8.4.12, verifying that the two graphs are isomorphic.

Exercise 10.3.3

Problem: Prove theorem 10.3.1.

Solution: If G is a graph such that there exists a rotation p where every induced circuit is length three then G is a graph of triangles. As from every v_i there must exist such a circuit:

$$v_i \to v_j \to v_k \to v_i$$

Because we are not allowed loops and the circuit must be of length three. In such a graph every vertex is of degree greater than or equal to 2. Suppose for the sake of contradiction that $r(p)$ is not maximal, then since every circuit in $r(p)$ has length 3, there are $\frac{2m}{3}$ total circuits where m is the number of edges of G. To induce more than $\frac{2m}{3}$ circuits this would require the existence of a circuit of length less than three, being either length two or length one. Circuits of length are not possible as these would be loops, and a circuit of length 2 is not either since every vertex has degree greater than or equal to 2, which is a contradiction. Thus, $r(p)$ is maximal if p is a rotation of a graph G such that every induced circuit is length three.

Exercise 10.3.4

Problem: Prove theorem 10.3.2.

Solution: Let G be a bipartite graph with m edges such that p is a rotation such that all circuits induced by p are length four. Suppose for the sake of contradiction that $r(p)$ is not maximal, then there exists a rotation p' of G inducing $\frac{2m}{x} > \frac{2m}{4}$ circuits, which implies the average length of the circuits induced by p' is $x < 4$, therefore there exists a circuit of length one, two, or three induced by p'.

- Case 1: A circuit of length one would be a self-loop which is not allowed.
- Case 2: A circuit of length two. If every circuit induced by p is length four, then either zero circuits are induced and it holds vacuously, in which case $r(p)$ is maximal, or there exists at least one circuit of length four, and hence at least two edges and four vertices in G. In this scenario, all vertices of G have degree greater than or equal to two, contradicting that a circuit of length two is possible.
- Case 3: A circuit of length three is not possible since as G is bipartite after traveling along three edges it is not possible to end where you began.

Therefore, we have a contradiction in all three cases and thus $r(p)$ is maximal.

Exercise 10.3.5

Problem: Show that if a scheme for a rotation p of a graph G satisfies the following

Rule Q^*. If in row $i. \dots j, k \dots,$
and in row $k. \dots i, \ell, \dots,$
then in row $j. \dots \ell, i \dots,$
and in row $\ell. \dots k, j, \dots,$
then every circuit induced by p has length four.

Solution: Any circuit induced by the rotation p satisfying Rule Q^* can be read off the scheme as follows. If you enter vertex i from vertex j then exit to vertex k. If you enter vertex k from vertex i then exit to vertex ℓ. If you enter vertex ℓ from vertex k then exit to vertex j. If you enter vertex j from vertex ℓ then exit to vertex i. If you enter vertex i from vertex j then exit to vertex k.... One can then see that every circuit induced by p has length four, as we just saw that any arbitrary circuit is length four, more specifically we traced the path of the circuit:

$$j \to i \to k \to \ell \to\to j \to i \to k \to \ell \dots$$

Exercise 10.3.6

Problem: Show that Rules Δ^* and R^* are equivalent. (That is, show that a scheme satisfies Rule Δ^* if and only if it satisfies Rule R^*.)

Solution:

Rule Δ^* says
If in row $i. \dots j, k \dots,$
then in row $k. \dots i, j, \dots,$
Rule R^* says
If in row $i. \dots j, k, \ell \dots,$
then in row $k. \dots \ell, i, j, \dots,$
First, we show that if a scheme satisfies Rule R^* then it satisfies Rule Δ^*. Simply omitting the ℓ in row i and row k we see that any scheme satisfying Rule R^* satisfies Rule Δ^*. Now suppose a scheme satisfies Rule Δ^* and suppose the entry after k in row i is ℓ, then we have:

$$i. \dots j, k, \ell, \dots$$

Since the scheme satisfies Rule Δ^* we have that in row ℓ

$$\ell. \dots i, k, \dots$$

Again because the scheme satisfies Rule Δ^* we have that in row k

$$k. \dots \ell, i, \dots$$

Which shows that we satisfy Rule R^*, since if in row i

$$i. \dots j, k, \ell, \dots$$

Then we have shown in row k

$$k. \dots \ell, i, j, \dots$$

Exercise 10.3.7

Problem: Find the genus of the 6-cage of Figure 4.2.4.

Solution: The genus is given by the non negative integer solution g to the equation $14-21+r(p) = 2-2g$ where $r(p)$ is the amount of circuits induced by a maximal rotation p of G. By theorem 10.3.2 if there was a rotation p inducing circuits all of length four then p would be maximal as G is bipartite. However, G has 21 edges so if all circuits were of length four then we would have $\frac{42}{4} = 10.5$ circuits which is not possible. The scheme:

1. 2 14 6
2. 11 3 1
3. 2 4 8
4. 3 13 5
5. 6 4 10
6. 5 1 7
7. 8 6 12
8. 7 9 3
9. 8 10 14
10. 9 11 5
11. 10 12 2
12. 11 7 13
13. 12 14 4
14. 13 1 9

Induces 7 circuits all of length 6. We also know by theorem 10.1.2 that the sum $p - q + r(p) = 14 - 21 + r(p)$ is even, thus a rotation inducing 10 or 8 circuits is not possible and a maximal rotation p of G induces either 9 or 7 circuits. Inducing 9 circuits would require three of the circuits to have length six and six of the circuits to have length four. This is not possible as the shortest cycle in a 6-cage is length 6. Therefore, we see that $g = 1$ satisfies $14 - 21 + 7 = 2 - 2g$ and thus the genus of the 6-cage of Figure 4.2.4 is 1.

Exercise 10.3.8

Problem: Determine a lower bound for the genus of the 10-cage of Figure 4.2.8.

Solution: The 10-cage in Figure 4.2.8 is Balaban's 10-cage with 70 vertices and 105 edges. Due to the equation $p - q + r(p) = 2 - 2g$ needing a non-negative integer solution g we can see that $g = 0$ is not possible as that would require the existence of a rotation p inducing 37 circuits which would have average length $\frac{210}{37} \approx 5.67..$ however the shortest cycle in a 10-cage is length 10 so this is not possible. Likewise, $g = 1, 2, 3, 4, 5, 6, 7$ are all not possible by the same argument. Therefore a lower bound for the genus of the 10-cage of Figure 4.2.8 is $g \geq 8$.

Exercise 10.3.9

Problem: Using Theorem 10.3.6, determine n so that $\gamma(K_n) = n$.

Solution: By Theorem 10.3.6 $\gamma(K_{18}) = \left\lceil \frac{(19-3)(19-4)}{12} \right\rceil = 18$

Exercise 10.3.10

Problem: Using Theorem 10.3.6, determine n so that $\gamma(K_n) = n + 1$.

Solution: By Theorem 10.3.6 $\gamma(K_{19}) = \left\lceil \frac{(19-3)(19-4)}{12} \right\rceil = 20$